# Propositional Logic: Symbols and Translations

In this lesson, we’ll begin our study of propositional logic. **Propositional logic** is a form of deductive logic that studies the relationship between different statements (or “propositions”). Just as categorical logic allows us to prove things about how different *categories* relate to one another, propositional logic will allow us to see how the *truth* or *falsity* of one statement allows to make inferences about the truth or falsity of *another* statement.

1. What is propositional logic? What are the logical operators (or connectives)?
2. What is the difference between a simple statement (or simple proposition) and compound statement (or compound proposition)?
3. What symbols do we use to represent negation, conjunctions, disjunctions, material implication, and material equivalence?
4. What are the antecedent and consequent of a conditional statement? How do these relate to the concepts of sufficient condition and necessary condition?
5. What is the main operator of a compound statement? What does it mean to say that a certain statement is a well-formed formula (WFF)?

Propositional logic, like categorical logic, has a long history going all the way back to Greek philosophers. In particular, the Stoic philosopher **Chrysippus** (279-206 BCE) is often credited with an early formulation. However, serious study of it didn’t really start until the 17th-19th centuries, when mathematicians like **Gottfried Leibniz, George Boole,** and **Augustus DeMorgan** began working on formal representations of deductive reasoning. Over the last 150 years or so, logicians expanded the basic ideas of propositional logic to produce **predicate logic** (which combines aspects of both categorical and propositional logic). These ideas provided the basis for the development of computers, and much contemporary work on this sort of logic is done by computer scientists.

## Basic Concepts in Propositional Logic

Propositional logic involves **simple statements** (statements that don’t contain any other statements as parts) connected with logical operators, such as AND, OR, NOT, and IF-THEN. A **compound statement** is a statement that *does* contain other statements as parts.

**Simple statements are represented by capital letters.** “Wilbur is a pig” and “Babe is a pig” are both simple statements. In propositional logic, these statements are represented by upper case letters like W and B.

**Compound statements put these letters together.** “Both Wilbur and Babe are pigs” is a compound statement (since it contains two simple statements are parts, which it joins with the logical word “and”). You could represent this as “W AND B,” where AND here is a logical operator (see below).

**The Logical Operators.** Propositional logic uses **logical operators** (usually represented by symbols) to connect or modify simple statements: OR, XOR, AND, IF-THEN, and NOT. Most systems don’t use XOR; I’m including it here just so you can see the contrast it with OR (and because it is very close to the English word “or”)

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| Operator | Name | Symbols | Truth conditions |
| P OR Q | Disjunction |  | True when either P is true, Q is true, or both are true. False only when BOTH P and Q are false. (So, “either Lincoln was a US President or Washington was” is TRUE.) |
| P XOR Q | Exclusive Disjunction | ------ | True when either P is true or Q is true. False when both are false or both are true. This corresponds to the English word “or”, but is NOT part of most propositional logic systems. |
| NOT P | Negation |  | True whenever P is false. means the same thing as P. (So, “it is false that the Yankees didn’t win” means “The Yankees won.”) |
| IF P THEN Q | Material Implication |  | False ONLY when the **antecedent** P is true and the **consequent** Q is false, and true in every other case. We say that P is **sufficient** for Q, and that Q is **necessary** for P. |
| P AND Q | Conjunction |  | True when both P is true and Q is true; false in every other case. |
| P EQUALS Q | Material Equivalence |  | True whenever P and Q have the *same* truth value (both are true, or both are false). Importantly, this does NOT mean that P and Q “mean the same thing.” It just means they are either both true, or both false. |

## Translating Statements into Propositional Logic

In order to use propositional logic to evaluate arguments written in English, we need to first translate them into the appropriate symbols. When doing this, it is important to remember that you’re never going to be able to capture *everything* about an English statement in propositional logic. In particular, we can’t always capture the **connotation** of sentences. For example, the English statement “I really wanted to eat doughnuts, but I ate salad instead” becomes something like “I wanted to eat doughnuts AND I did not eat doughnuts AND I ate salad.” This obviously misses part of the “feel” of the original statement.

Here are some “tips” for translating common expressions:

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| English Expression | Translation |
| “It is false that P”, “Not P”, “It is not the case that P” | ~P |
| “P and Q”, “P but Q”, “P; however, Q”, “Both P and Q”, “P; additionally, Q” | P ∙ Q |
| “P or Q”, “P unless Q” | P ∨ Q |
| “If P then Q”, “P only if Q”, “Q if P”, “Given that P, Q”, “P is a sufficient condition for Q”, “Q is a necessary condition for P” | P ⊃ Q |
| “P if and only if Q”, “P is equivalent to Q”, “P is a necessary and sufficient condition for Q” | P ≡ Q |
| “Neither P nor Q”, “Both P and Q are false” | ~P ∙ ~Q or ~(P ∨ Q) |

## Main Operators and Well-Formed Formulas

Just as is the case with categorical logic (and just as is the case with ordinary English) statements in propositional logic have their own “grammar” that you’ll need to follow. A statement that follows this grammar is called a **well-formed formulas (wff)**. If a statement is NOT a well-formed formula, then it is (quite literally) “meaningless” from the standpoint of propositional logic. The rules for wffs are quite strict, and admit of no exceptions. The reason for this is because we want to avoid any possibility of ambiguity or misinterpretation: we need to make it *perfectly clear* which operators apply to which things.

When translating English statements into propositional logic, you need to remember some simple rules in order to make sure you are producing

1. Negation (tilde) applies only ONE statement. So, ~P is a wff. However, S~P is not. When a tilde is outside a parentheses, as in , it applies to EVERYTHING inside the parentheses.
2. All of the other operators ( take exactly two statements (one on each side). So, is a wff, but is not.
3. You need **parentheses** around each pair of statements if you have more than one operator (tildes don’t count for this). So is a wff, but is not. However, is fine.

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| Examples: Well-formed formulas | |  | Examples: Not well-formed | |
| ~P | ~P ≡ P | P~ | P ~≡ P |
| ~~P | P ⊃ (Q ∙ ~R) | ~P~ | P ⊃ (Q ∙ ~R |
| P v Q | R ⊃ Q | v(PQ) | RR ⊃ Q |
| P ∙ ~Q | [R ⊃ (Q ⊃ P)] ∙ Z | P ∙ Q~ | [R ⊃ (Q ⊃ P) ∙ Z] |
| ~(P ∙ Q) | (P ∙ Q) v R | ~P ∙ Q) | P ∙ Q v R |

**Where is the main operator?** Every compound statement has exactly one **main operator** that determines the truth value of the whole statement. It is the operator whose truth value comes “last” when you are working out the statement’s truth value following the “order of operations.”

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| **Examples: Finding the Main Operator** | | | |
| ~P | ~~P | P ∙ (Q ∙ R) | (P v Q) ∙ (Q ⊃ P) |
| P ⊃ Q | ~P ∙ ~Q | (P ∨ Q) ≡ R | [(P v Q) ∙ (Q ⊃ P)] ⊃ R |
| ~P ⊃ Q | ~(P ∙ Q) | ~(P ∨ Q) ≡ R | ~R ⊃ (P ∙ Q) |
| ~(P ⊃ Q) | ~(~P ≡ Q) | ~[(P ∨ Q) ≡ R] | ~(P ≡ R) ≡ T |

## Solved problems

Problem 1. Suppose that **P = Abraham Lincoln was a U.S. president and W = Abraham Lincoln was a werewolf.** Use this to determine the (1) meaning and (2) truth value of the following compound statements.

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| Statement | Means | True? |
| P | Abe Lincoln (“Abe”) was a U.S. President | True |
| W | Abe was a werewolf. | False |
| P ∙ W | Abe was both president and a werewolf. | False |
| W~ | ? | This isn’t a WFF |
| P v W | Abe was either president or a werewolf | True |
| P ≡ ~W | Abe was president if and only if he wasn’t a werewolf | True |
| P ⊃ W | If Abe was president, he was a werewolf | False |
| S v ~S | Huh? What does S mean? | True! (This says “Either S is true or it is false.” As it turns out, this is always true, regardless of what S means). |
| W ⊃ P | If Abe was a werewolf, he was president. | True! (This one is weird, but ANY conditional with a false antecedent is true). |
| W P | ? | This isn’t a WFF |
| W ∙ W | Abe was a werewolf and Abe was a werewolf | False. (Another weird one. This actually is a wff; it means the exact same thing as W). |
| ~P ∙ W | Abe wasn’t president, but was a werewolf | False. |
| S ∙ ~S | Here’s S again—we don’t know what this means. | False (No matter was S means, it can’t be true and false at the same time) |
| ~(P ∙ W) | It is false that Abe was BOTH president and a werewolf. | True. (See how parentheses make a difference?) |
| ~(P ∙ W) v P | Either it is false that Abe was BOTH president and a werewolf, or he was president. | True. (Here, we’re finally getting to a statement that our brains can’t deal with very well. This is where propositional logic starts coming in handy.) |
| ~[~(P ∙ W) v P] | The statement on the line above is false. | False. We know this because the negation in the front (the main operator of this statement) just reverses the truth value of whatever it applies to. |
| W ≡ ~[~(P ∙ W) v P] | W has the same truth value as the statement on the line above. | True (W is false, and so was the statement on the line above). |
| ~[~P ∙ W v P] | ? | Not a WFF |

## Review Questions

1. Translate the following statements into propositional logic. Suppose that S = “Susan will be at the party,” P = “Percy will be at the party,” and T = “Theo will be at the party.” Then, see if you can identify the main operator.
   1. It is false that Susan will be at the party.
   2. Susan and Percy will both attend the party.
   3. Either Susan will be at the party, or Percy will be, or both will be.
   4. Susan will be at the party if Percy is.
   5. If Susan is not at the party, then Percy won’t come either.
   6. Neither Susan nor Percy will be at the party.
   7. Susan will either be at the party, or she won’t be.
   8. If Percy and Susan are both at the party, then it is true that either Percy or Susan are at the party.
   9. Theo will be at the party if and only either Susan is there or Percy is not.
   10. If Theo comes to the party, then Percy will come as well, unless Susan does not come.
2. Determine the truth of the following conditional statements.
   1. If the Earth is a planet, then the Earth is not flat.
   2. If the Earth is flat, the Earth is a planet.
   3. If the Earth is flat, the Earth is not a planet.
   4. If the Earth is not a planet, then the earth is flat.
   5. If the earth is not flat, then the Earth is not a planet.
   6. If the earth is not flat, then the Earth is a planet.
3. Write down one simple statement that is false and one simple statement that is true. Now, use these to create the following:
   1. A compound statement with 1-2 operators that is true.
   2. A compound statement with 1-2 operators that is false.
   3. A compound statement that is not well-formed.
   4. A compound statement with 3+ operators that is true.